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SINGULARITY AND MESH DIVERGENCE OF INVISCID ADJOINT SOLUTIONS AT SOLID WALLS

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1 INTRODUCTION

The aim of this paper is to describe a problem for the adjoint Euler equations that is best illustrated with a simple example: the inviscid incompressible flow at angle of attack $\alpha = 0°$ past a symmetrical van de Vooren airfoil with 12% thickness and trailing edge angle $\tau = 16°$ built from the circle $\zeta = Re^{i\theta}$, $0 \le \theta < 2\pi$, via the conformal transformation [1]

$$z(\zeta) = \frac{(\zeta - R)^k}{(\zeta - \varepsilon R)^{k-1}} + 1 \tag{1}$$

with $R = (1+\varepsilon)^{k-1} / 2^k$, $\varepsilon = 0.0371$ and $k = 86/45$. The trailing edge is at $z = 1$, which corresponds to $\zeta_{te} = X_{te} + iY_{te} = (R, 0)$ in the circle plane. Figure 1 shows the adjoint values on the airfoil profile across several mesh levels computed with the SU2 incompressible solver [2]. It is observed that the drag-based adjoint solution (left) behaves smoothly and converges with mesh refinement, while the lift-based adjoint solution (right) diverges at the trailing edge on any given mesh and the value along the remainder of the airfoil grows continually as the mesh density increases.

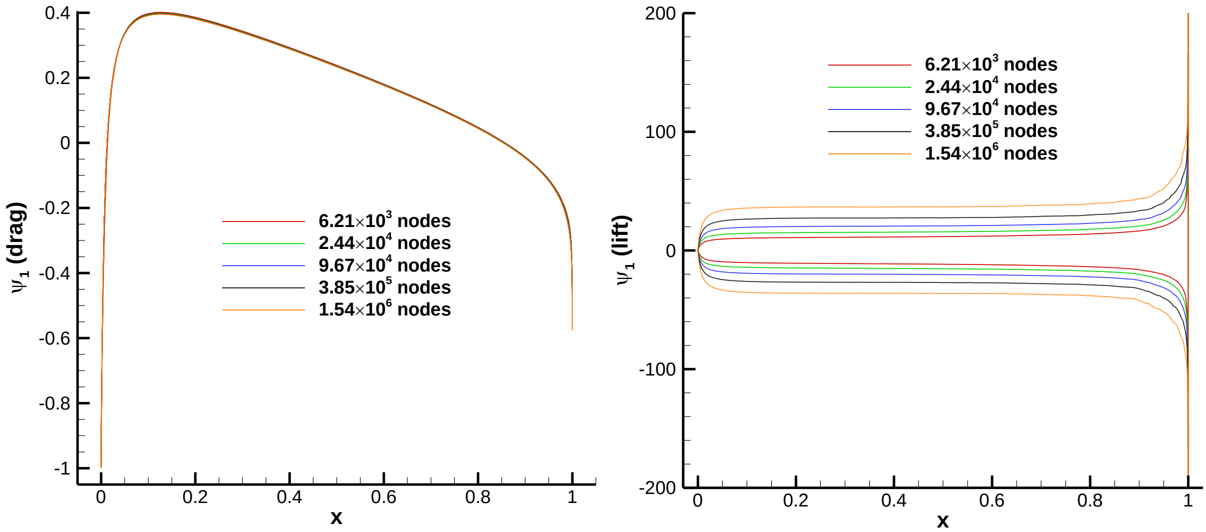

Figure 1. Drag (left) and lift (right)-based inviscid incompressible adjoint solution on the van de Vooren airfoil profile (1) at α = 0° computed with the SU2 solver on 5 progressively refined unstructured triangular meshes.

2 CHARACTERIZATION OF THE MESH DIVERGENCE PROBLEM

In order to characterize the problem, several tests have been performed [3] [4], with the following conclusions:

1. The problem is limited to inviscid cases.
2. The adjoint-based sensitivity derivatives are not affected.
3. The issue depends on the cost function and flow regime as follows:
   - In supersonic flow, lift or drag-based adjoint solutions do not show this behavior.
   - In transonic and subsonic flow, including incompressible flow, lift-based adjoint solutions are always affected, while drag-based solutions are only affected for transonic rotational flows (e.g. shocked flow past a symmetric airfoil with non-zero angle of attack).
   - The adjoint state based on the far-field entropy flux shows the same behavior as the near-field drag.
4. The problem is not exclusive of a particular solver or numerical scheme, having been observed with wildly different solvers (DLR's Tau code [5], Stanford University's SU2 code [2], ONERA's ELSA code [6] and Imperial College's Nektar++ code [7]).
5. The issue appears in two and three dimensions.
6. The adjoint wall boundary conditions is reasonably well obeyed across mesh levels except in the proximity of the trailing edge.
7. The anomaly is observed in all types of trailing edge configurations (including blunt and cusped trailing edges) and also in blunt bodies such as circles and ellipses.
8. The issue does not seem to depend on the far-field distance, resolution or the adjoint far-field b.c.
9. The anomaly is tied to adjoint singularities at the trailing edge or rear stagnation point and at the incoming stagnation streamline.
10. It was shown that increasing dissipation levels did not prevent the mesh divergence, but the actual value of the adjoint solution at the wall on a given mesh was found to depend strongly on the dissipation level.
11. Finally, it was shown in [6] that linear perturbations to lift or drag caused by numerical solutions containing point sources corresponding to stagnation pressure perturbations do appear to diverge towards the wall.

3 ANALYTIC ADJOINT SOLUTION FOR INCOMPRESSIBLE FLOW

Item 11 above hints at an adjoint singularity at the wall of the same nature as the well-known singularity along the incoming stagnation streamline discovered in [8]. An adjoint singularity at the wall would certainly explain the observed behavior of numerical solutions, but it would remain to determine the origin and characteristics of the singularity and to explain, likewise, how a singular (i.e. infinite) adjoint solution could be reconciled with the adjoint wall b.c. and sensitivity derivatives.

Fortunately, an analytic solution for the lift and drag-based adjoint two-dimensional incompressible Euler equations was obtained in [9] using the Green's function approach [10]. The resulting drag-based adjoint solution is smooth, but the lift-based adjoint solution contains singularities at the trailing edge, due to the sensitivity of the Kutta condition to perturbations of the flow, and also along the incoming stagnation streamline and the wall (see Figure 2 and Figure 3). Further, it can be shown that two particular linear combinations of adjoint variables, which yield the continuous adjoint sensitivity derivatives and the adjoint wall boundary conditions, respectively, are actually free of singularities.

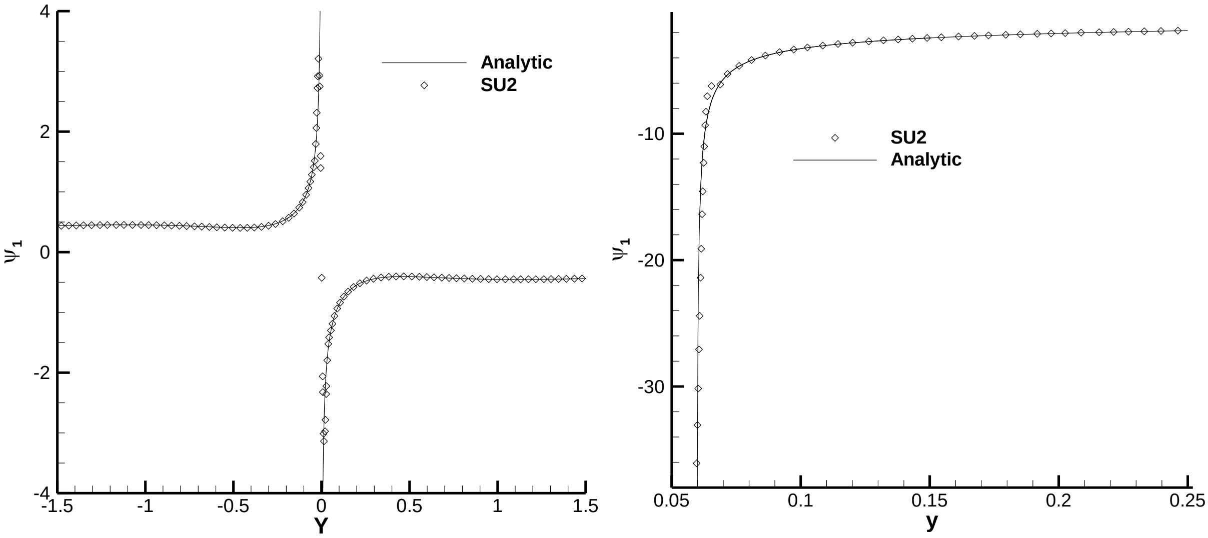

Figure 2. ψ_1 along lines crossing the stagnation streamline upstream of the airfoil (left) and normal to the airfoil wall at $x/c = 0.31$ (right).

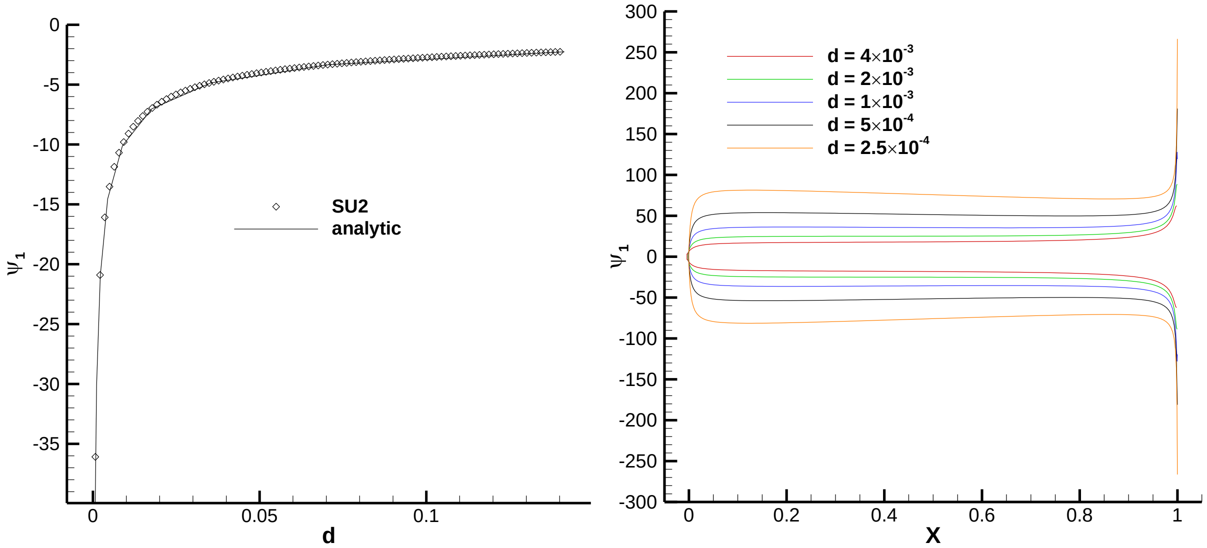

Figure 3. Lift-based ψ_1 along a line approching the trailing edge (left) and on a sequence of O-shaped curves surrounding the airfoil (right). The O-curves are built in the circle plane as circumferences concentric with the circle and are transferred to the airfoil plane with (1).

4 CONCLUSIONS

The near-wall mesh divergence of solutions to the adjoint Euler equations occurring at subsonic and transonic speeds is reviewed. By examining a recently derived analytic adjoint solution, it is shown that the anomaly observed in numerical computations is caused by a divergence of the analytic solution at the wall. On the numerical side, the numerical viscosity of the solver stabilizes the divergence, producing a finite value at the wall which nevertheless varies continually as the grid spacing or the intensity of the numerical dissipation change

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